HFTA-05.0: Statistical Confidence Levels for Estimating BER Probability

When testing for bit error ratio (BER), how many bits do you need to transmit through a system in order achieve reasonable confidence in the results? This article provides the answer. It starts with the theory of statistical confidence levels and then shows how to apply the theory to the BER measurement problem. It finishes up with clear examples and step-by-step instructions for setting up the appropriate tests.

Many components in digital communication systems must meet a minimum specification for the probability of bit error (P(ε)). For a given system, P(ε) can be estimated by comparing the output bit pattern with a predefined pattern applied to the input. Any discrepancy between the input and output bit streams is flagged as an error, and the ratio of detected bit errors (ε) to total bits transmitted (n) is P'(ε), where the prime character signifies an estimate of the actual P(ε). The quality of this estimate improves with the total number of bits transmitted. The relationship can be expressed as

$$P'(\varepsilon) = \frac{\varepsilon}{n} \xrightarrow[n \to \infty]{} P(\varepsilon)$$ [eq. 1]

It is important to transmit enough bits through the system to ensure that P'(ε) is a reasonable approximation of the actual P(ε) (i.e., the value to be obtained if the test could proceed for an infinite amount of time). For a reasonable limit on test time, therefore, we must know the minimum number of bits that yields a statistically valid test.

In many cases, we must verify only that P(ε) is at least as good as some predefined standard. In other words, it is sufficient to prove that P(ε) is less than some upper limit. For example, the P(ε) required in many telecommunication systems is 10^{-10} or better (an upper limit of 10^{-10}). The statistical idea of associating a *confidence level* with an upper limit can be used to postulate, with quantifiable confidence, that the actual P(ε) is less than the specified limit. As a primary advantage, this method lets you trade test time for measurement accuracy.

Defining the statistical confidence level

The statistical confidence level is defined as the probability, based on a set of measurements, that the actual probability of an event is better than some specified level. (For the purpose of this definition, actual probability means the probability that is measured in the limit as the number of trials tends toward infinity.) When applied to P(ε) estimation, the definition of statistical confidence level can be restated as the probability (based on e detected errors out of n bits transmitted) that the actual P(ε) is better than a specified level γ (such as 10^{-10}).

Mathematically, this can be expressed as

$$CL = P\left[P(\varepsilon) < \gamma \middle| \varepsilon, n\right]$$ [eq. 2]

where P[ ] indicates probability and CL is the confidence level. Because confidence level is a probability by definition, the possible values range from 0% to 100%.

After computing the confidence level, we can say we have CL percent confidence that the P(ε) is better than γ. As another interpretation, if we repeat the bit-error test many times and recompute P'(ε) = ε/n for each test period, we expect P'(ε) to be better than γ for CL percent of the measurements.

Calculating the confidence level

Calculations of the confidence level are based on the binomial distribution function described in many statistics texts[1,2]. The binomial distribution function is generally written as

$$P_n(k)=\binom{n}{k}p^k q^{n-k}, \text{ where } \binom{n}{k} \text{ is defined as } \frac{n!}{k!\,(n-k)!} \quad \text{[eq. 3]}$$

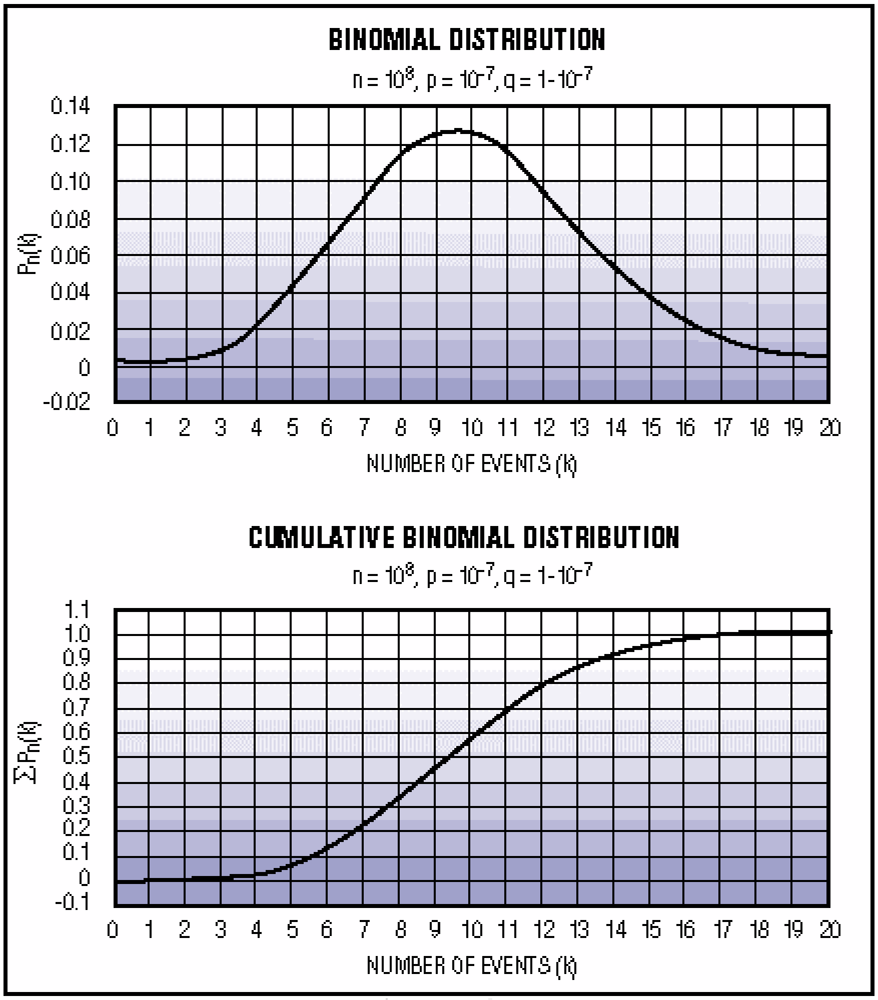
Equation [3] gives the probability that k events (i.e., bit errors) occur in n trials (i.e., n bits transmitted), where p is the probability of event occurrence in a single trial (i.e., a bit error), and q is the probability that the event does not occur in a single trial (i.e., no bit error). Note that the binomial distribution models events that have two possible outcomes, such as success/failure, heads/tails, or error/no error. Thus, $p+q=1$.

When we are interested in the probability that N or fewer events occur in n trials (or, conversely, that greater than N events occur), then the cumulative binomial distribution function of Equation 4 is useful:

$$P(\varepsilon \le N)=\sum_{k=0}^{N} P_n(k)=\sum_{k=0}^{N}\left(\frac{n!}{k!\,(n-k)!}\right)p^k q^{n-k}$$

$$P(\varepsilon > N)=1-P(\varepsilon \le N)=\sum_{k=N+1}^{n}\left(\frac{n!}{k!\,(n-k)!}\right)p^k q^{n-k} \quad \text{[eq. 4]}$$

Graphical representations of Equations 3 and 4, along with some of their properties, are summarized in **Figure 1**.



$$P_n(k) = \left(\frac{n!}{k!\,(n-k)!}\right) p^k q^{n-k}$$

n = Total number of trials (i.e., total bits transmitted)

k = Number of events occurring in n trials (i.e., bit errors)

p = Probability that an event occurs in one trial (i.e., probability of bit error)

q = Probability that an event does not occur in one trial (i.e., probability of no bit error)

$p + q = 1$
Mean (μ) = nq
Variance (σ^2) = npq

$$CL = 1 - \sum_{k=0}^{N} P_n(k)$$

Figure 1. The binomial and cumulative binomial distributions relate number of trials and measured error to the probabilities that an error will (or will not) occur.

Binomial distribution function

In a typical confidence-level measurement, we start by choosing a satisfactory level of confidence and hypothesizing a value for p (the probability of bit error in transmitting a single bit). We represent the chosen p value as p_h. In general, we choose these values according to a limit imposed by specification (e.g., if the specification is $P(\varepsilon) \le 10^{-10}$, we choose $p_h = 10^{-10}$ and a confidence level of, say, 99%).

We can then use Equation 4 to determine the probability $P(\varepsilon > N|p_h)$, based on p_h, that more than N bit errors will occur when n total bits are transmitted. If, during actual testing, less than N bit errors occur (even though $P(\varepsilon > N|p_h)$ is high), then one of two conclusions can be made: (a) we just got lucky, or (b) the actual value of p is less than p_h. If we repeat the test over and over and continue to measure less than N bit errors, then we become more and more confident in conclusion (b).

The quantity $P(\varepsilon > N|p_h)$ defines our level of confidence in conclusion (b). If $p_h = p$, we have a high probability of detecting more bit errors than N. When we measure less than N errors, we conclude that p is probably less than ph, and we define as the confidence level this probability that our conclusion is correct. In other words, we are CL% confident that $P(\varepsilon)$ (the actual probability of bit error) is less than p_h.
In terms of the cumulative binomial distribution function, the confidence level is defined as

$$CL = P\left(\varepsilon > N \;\middle|\; p_h\right) = 1 - \sum_{k=0}^{N} \left(\frac{n!}{k!\,(n-k)!}\right) (p_h)^k (1-p_h)^{n-k} \quad \text{[eq. 5]}$$

where CL is the confidence level in terms of percent.

As noted above, when using the confidence-level method we generally choose a hypothetical value of p (p_h) along with a desired confidence level (CL), and then solve Equation 5 to determine how many bits (n) we must transmit through the system (with N or fewer errors) to prove our hypothesis. Solving for n and N can prove difficult unless approximations are made.

If np > 1 (i.e., we transmit at least as many bits as the mathematical inverse of the bit error rate), and k has the same order of magnitude as np, then the Poisson theorem(1) (Equation 6) provides a conservative estimate of the binomial distribution function:

$$P_n(k) = \left(\frac{n!}{k!\,(n-k)!}\right) p^k q^{n-k} \xrightarrow[n \to \infty]{} \frac{(np)^k}{k!} e^{-np} \quad \text{[eq. 6]}$$

Equation 7 shows how Equation 6 can be used to obtain an approximation for the cumulative binomial distribution as well:

$$\sum_{k=0}^{N} P_n(k) = \sum_{k=0}^{N} \frac{(np)^k}{k!} e^{-np} \quad \text{[eq. 7]}$$

We can combine Equations 5 and 7, and solve for n as follows:

$$\sum_{k=0}^{N} P_n(k) = 1 - CL$$

(by rearranging Equation 5)

$$\sum_{k=0}^{N} \frac{(np)^k}{k!} e^{-np} = 1 - CL$$

(using Equation 7)

$$-np = \ln\left[\frac{1-CL}{\sum_{k=0}^{N}\frac{(np)^k}{k!}}\right] \quad n = -\frac{\ln(1-CL)}{p} + \frac{\ln\left(\sum_{k=0}^{N}\frac{(np)^k}{k!}\right)}{p}$$

[eq. 8]

Note that the second term in Equation 8 equals zero for N = 0, and for that case the equation is easily solved. Solutions to Equation 8 are more difficult for N > 0, but they can be obtained empirically, using a computer. We can now determine the total number of bits that must be transmitted through the system to achieve a desired confidence level. Following is an example of this procedure:

1) Select p_h, the hypothetical value of p. This value is the probability of bit error that we would like to verify. For example, if we want to show that $P(\varepsilon) \leq 10^{-10}$, then we set p in Equation 8 equal to $p_h = 10^{-10}$.
2) Select the desired confidence level. Here we are forced to trade confidence for test time. To minimize test time, choose the lowest reasonable confidence level. The trade-off between test time and confidence level is proportional to -ln(1 - CL). See **Figure 2**.
3) Solve Equation 8 for n. In most cases, this task is simplified by assuming that no bit errors will occur during the test (i.e., N = 0).
4) Calculate the test time. The time required to complete the test is n/R, where R is the data rate.

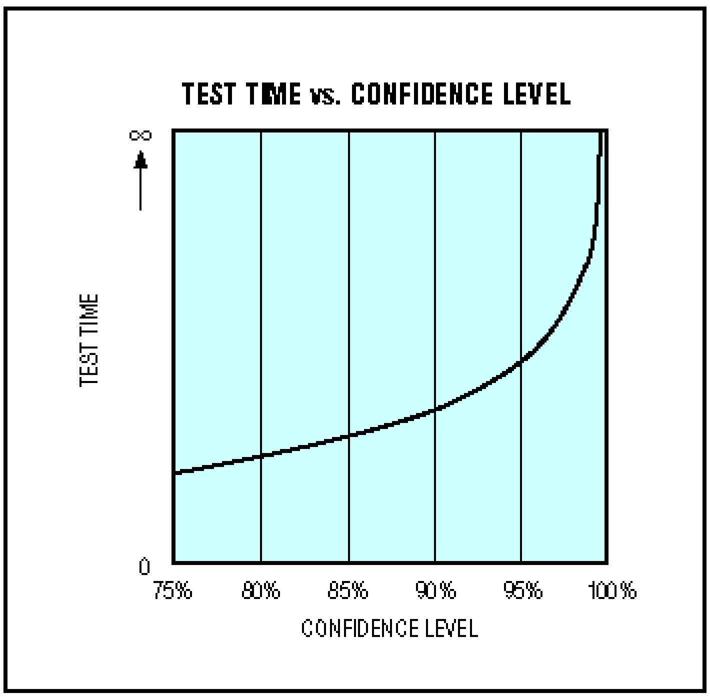


Figure 2. Confidence level (in a specified error rate) rises with the allowed test time.

Using CL to estimate P(ε)

Many telecommunication systems specify 10^{-10} or better for $P(\varepsilon)$. Assume that we must test two clock/data-recovery chips, the MAX3675 (622Mbps) and the MAX3875 (2.5Gbps), to verify compliance with this specification. We first set $p_h = 10^{-10}$. We would like a test that yields 100% confidence in the desired specification, but that requires an infinite test time. We therefore settle for a confidence level of 99%. Next we solve Equation 8 for n using various values of N (0, 1, 2, 3, etc.). The results are shown below in **Table 1**.

Table 1. Estimation of Bit Error Probability (Example: CL = 99% and $p_h = 10^{-10}$)

Bit Errors ≤ N N=	Required Number of Bits to Transmit (n)	Test Time for Bit Rate of 622Mbps (seconds)	Test Time for Bit Rate of 2.5Gbps (seconds)
0	$4.61 \cdot 10^{10}$	74.1	18.5
1	$6.64 \cdot 10^{10}$	106	26.7
2	$8.40 \cdot 10^{10}$	135	33.7
3	$1.00 \cdot 10^{11}$	161	40.2
4	$1.16 \cdot 10^{11}$	186	46.6

From Table 1 we see that if no bit errors are detected for 18.5s (in a 2.5Gbps system), then we have a 99% confidence level that $P(\varepsilon) \leq 10^{-10}$. If one bit error occurs in 26.7s of testing, or two bit errors in 33.7s, the result is the same: a 99% confidence level that $P(\varepsilon) \leq 10^{-10}$.

To develop a standard $P(\varepsilon)$ test for the MAX3675/ MAX3875, we might select the test time corresponding to N = 3 from Table 1. Using a bit-error-rate tester (BERT), we transmit 10^{11} bits through each of the two chips. The test time for 10^{11} bits is 2min 41s at 622Mbps, or 40.2s at 2.5Gbps. At the end of the test time, we check the number of detected bit errors (ε). If $\varepsilon \leq 3$, the device has passed and we are 99% confident that $P(\varepsilon) \leq 10^{-10}$.

Stressing the system to reduce test time

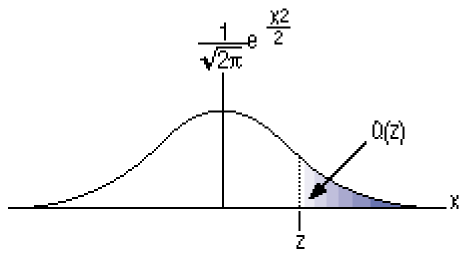
Dan Wolaver has documented a method for reducing test time by stressing the system(3). It is based on an assumption that the dominant cause of bit errors is thermal (Gaussian) noise at the input of the receiver. (Note that this assumption excludes jitter and other potential causes of error.) For systems in which this assumption is valid, the signal-to-noise ratio (SNR) can be reduced by inserting a fixed attenuation in the transmission path (i.e., the attenuation applies to the signal only; not the dominant noise source). In the previous example (MAX3675 and MAX3875), it was determined that jitter effects and nonlinear gain in the input limiting amplifier violated the key assumptions of this method, so it was not employed.
In systems where the assumption is valid, the probability of bit error can generally be calculated(4, 5) as:

$$P(\varepsilon) = Q\left(\frac{\sqrt{SNR_{electrical}}}{2}\right) = Q\left(\frac{\sqrt{SNR_{optional}}}{2}\right) \quad \text{[eq. 9]}$$

where Q(x) is the complementary error (or the "Q" function included in many communications textbooks(6). A variety of other sources for this data are available, including the NORMDIST function in Microsoft Excel. Key values for the complementary error function are listed in **Table 2**.

Table 2. Tabulated Values for the Complementary Error ("Q") Function

$z = \frac{x - \mu}{\sigma}$	$Q(z) = \frac{1}{\sqrt{2p}} \int_z^{\infty} e^{\frac{x^2}{2}} dx$
3.71	10^{-4}
4.26	10^{-5}
4.75	10^{-6}
5.19	10^{-7}
5.61	10^{-8}
5.99	10^{-9}
6.36	10^{-10}
6.70	10^{-11}
7.03	10^{-12}



Equation 9 shows that the probability of bit error increases as the SNR decreases. If a fixed attenuation (α) is inserted in the transmission path, then the signal power (PS) is reduced by the factor α while the noise power (PN) is unchanged. The SNR is therefore reduced from SNR = PS/PN to SNR = PS/αPN. The corresponding P(ε) is increased by a factor that can be calculated using Equation 9 and Table 2.
We can now repeat the earlier test method using a modified value for p_h. The calculation can then be extrapolated to any other SNR by using Equation 9. The result is the same, but the test time may be significantly shorter.

The disadvantage of stressing a system is that measurements and calculations must be carried out with more precision, because extrapolating the results to their non-stressed levels multiplies the errors due to roundoff truncation, measurement tolerance, etc.

References

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A similar article appeared in the April 2000 issue of *Lightwave*.

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